Inverse Water Filling and the Sampling Theorem

Krish Eswaran

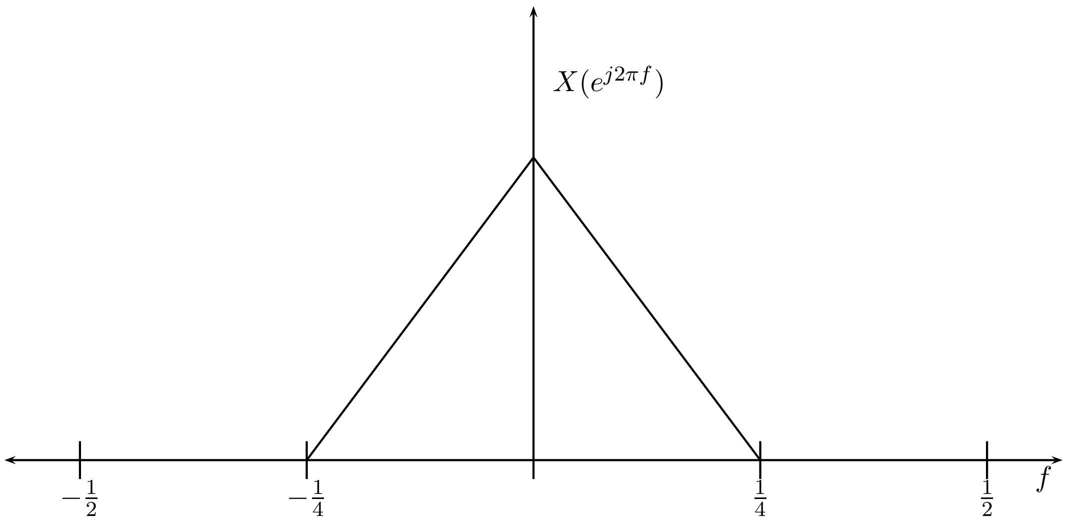

Fig. 1. A plot of the DTFT $X(e^{j2\pi f})$ of a signal $x[n]$ sampled at twice the Nyquist rate.

I. Introduction

This set of notes is an attempt to clarify a relationship between the Shannon-Nyquist sampling theorem and inverse water filling solution to the Gaussian rate-distortion function. Section II reviews the sampling theorem, and Section III provides an extension to random signals. Section IV provides a review of the quadratic Gaussian rate-distortion theorem, and Section V draws a connection between this result and the sampling theorem.

II. The Sampling Theorem

Let $x(t)$ be a continuous-time signal with Fourier transform

$$\mathbf{X}(F) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi F t} dt \ . \tag{1}$$

The sampling theorem tells us that if there exists a $B > 0$ such that $\mathbf{X}(F) = 0$ for $|F| > B$, then we can reconstruct the original signal $x(t)$ from a discrete-time sampled signal $x[n] = x(nT)$ for any sampling interval $T < \frac{1}{2B}$, where $f_s = 2B$ is sometimes called the Nyquist rate.

The relationship between DTFT of $x[n]$ and the Fourier transform of the original signal is

$$X(e^{j2\pi f}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j2\pi f n} \tag{2}$$

$$= \frac{1}{T} \sum_{n=-\infty}^{\infty} \mathbf{X}\left(\frac{f-n}{T}\right) \ . \tag{3}$$

The derivation is given in Appendix A.

Suppose that if $T = \frac{1}{4B}$. That is, the sampling frequency is twice the Nyquist rate. In this case, the DTFT plotted over $-\frac{1}{2} \le f \le \frac{1}{2}$ (it is 1-periodic) will look similar to Figure 1.

In this scenario, one can reconstruct $x[n]$, and thus the original $x(t)$, from the downsampled $y[n] = x[2n]$.

III. Extension to Random Signals

It turns out the above result extends naturally to random signals if, instead of reconstructing the signal exactly, one is simply interested in reconstructing the signal such that the mean-squared error of the reconstructed signal with respect to the original is zero.

Let $X(t)$ be a zero-mean and stationary Gaussian random process with auto-correlation function $R(\tau) = \mathbb{E}[X(t)X(t-\tau)]$ and power-spectral density

$$\mathbf{S}(F) = \int_{-\infty}^{\infty} R(\tau)e^{-j2\pi F\tau}d\tau \ . \tag{4}$$

Then, if there exists a B such that $\mathbf{S}(F) = 0$ for $|F| > B$, one can reconstruct $X(t)$ with no mean-squared error distortion from the discrete-time sampled random process $X_n = X(nT)$, where $T < \frac{1}{2B}$, where X_n has auto-correlation function $R[k] = \mathbb{E}[X_n X_{n-k}] = R(kT)$ and power-spectral density

$$S(e^{j2\pi f}) = \sum_{n=-\infty}^{\infty} R[k]e^{-j2\pi fk} \ . \tag{5}$$

One can see this via the orthogonality principle, which implies that given an optimal reconstruction filter $a(t)$ (i.e. Wiener filter), we will have

$$\mathbb{E}[(X(t) - \sum_{n=-\infty}^{\infty} a(t-nT)X_n)X_k] = 0 \ . \tag{6}$$

However, evaluating the expectation and rearranging terms simply gives

$$R(t-kT) = \sum_{n=-\infty}^{\infty} a(t-nT)R[n-k] \ . \tag{7}$$

By taking the Fourier transform of both sides, we get that

$$\begin{aligned}
\mathbf{S}(F)e^{-j2\pi kFT} &= \mathbf{A}(F)\sum_{n=-\infty}^{\infty} R[n-k]e^{-j2\pi FnT} \\
&= \mathbf{A}(F)e^{-j2\pi kFT}S(e^{j2\pi FT}) \\
&= \frac{1}{T}\mathbf{A}(F)e^{-j2\pi kFT}\sum_{n=-\infty}^{\infty} \mathbf{S}(F-n/T)
\end{aligned} \tag{8}$$

where (8) follows from the relationship between the Fourier transform and DTFT derived in (3). Solving for $\mathbf{A}(F)$, we get that the optimal Wiener filter for this problem is just

$$\begin{aligned}
\mathbf{A}(F) &= T \cdot \frac{\mathbf{S}(F)}{\sum_{n=-\infty}^{\infty}\mathbf{S}(F-n/T)} \\
&= \begin{cases} T, & |F| \le B \\ 0, & |F| > B \end{cases}
\end{aligned} \tag{9}$$

where the last line follows from our assumption that the power spectral density is 0 outside this interval and T is large enough such that no aliasing occurs.

Thus, the Wiener filter is a low-pass filter that effectively gives the same power-spectral density to the reconstructed signal as the original signal. Since the orthogonality principle implies the mean-squared error is simply the difference between the variances of the original and reconstructed signal, the fact that their power spectral densities are the same implies the mean-squared error is 0.

IV. Rate-Distortion Theory: Quantized Signals and Inverse Water Filling

Let us know stick with discrete time. Consider a discrete-time Gaussian random process $\ldots, X_{-1}, X_0, X_1, \ldots$ with the power-spectral density depicted in Figure 3. The goal is now to reconstruct the original signal, but we now only have a finite number of bits to represent each sample. Thus, even perfect-reconstruction in the mean-squared error sense is too restrictive. Instead, we will aim for the more modest goal of reconstructing to within some distortion D:

$$\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}[(X_i - \hat{X}_i)^2] \leq D \ . \tag{10}$$

If the X_i are independent and identically distributed (i.i.d.), then the solution for this problem lies in the quadratic Gaussian rate-distortion theorem, which states that for sufficiently large N, there exists a vector quantizer that achieves the distortion D with a rate arbitrarily close to the quadratic Guassian rate-distortion function $R(D)$, which is defined as

$$R(D) = \frac{1}{2}\log\frac{\sigma^2}{D} \ , \tag{11}$$

where σ^2 is the variance of X_i. Furthermore, there does not exist a vector quantizer that can achieve lower rates than $R(D)$.

Note that for the i.i.d. case, the power spectral density is rather flat and uninteresting, as in Figure 2.

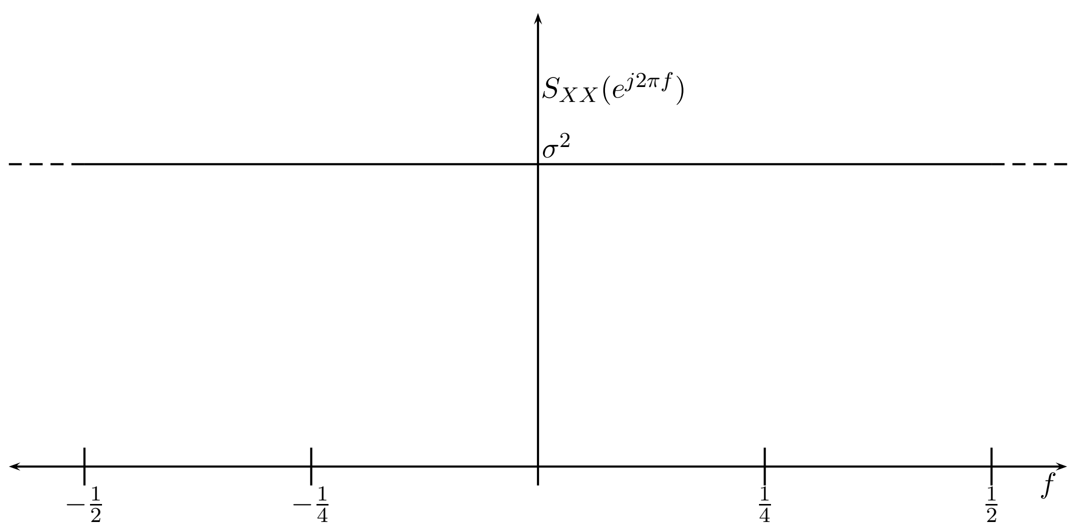

Fig. 2. Power-spectral density of a discrete-time i.i.d. Gaussian random process $\ldots, X_{-1}, X_0, X_1, \ldots$.

The key insight follows from noting that by multiplying an n-dimensional jointly Gaussian random vector by a matrix A yields a new Gaussian random vector with covariance matrix $AK_{XX}^{(n)}A^\dagger$. Thus, one can choose A to be the unitary matrix that diagonalizes the covariance matrix, thereby creating independent Gaussians with variances corresponding to the eigenvalues $\lambda_k^{(n)}$ of $K_{XX}^{(n)}$. Specifically, rearrange $X_1, \ldots, X_N$ into a matrix:

$$\mathbf{X} = \begin{bmatrix} X_1 & X_{n+1} & \ldots & X_{N-n+1} \\ X_2 & X_{n+2} & \ldots & X_{N-n+2} \\ \vdots & \vdots & \vdots & \vdots \\ X_n & X_{2n} & \ldots & X_N \end{bmatrix} \tag{12}$$

By defining $\mathbf{Y} = A\mathbf{X}$, we diagonalize the covariance matrix of every column of $\mathbf{Y}$. Then, by choosing an optimal vector quantizer with distortion D_k for row k, as $\frac{N}{n} \to \infty$, the rate for each row tends to

$$R_k = \frac{1}{2}\log\frac{\lambda_k^{(n)}}{D_k} \ , \tag{13}$$

Since unitary transforms preserve mean squared error, $D = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} D_k$, and the rate per sample is $\lim_{n\to\infty} n^{-1}\sum_{k=1}^{n} R_k$.

It remains to allocate the rates R_k for each independent transformed sources (or, equivalently, their distortions D_k) to give the best rate-distortion tradeoff. This follows from the inverse water filling solution, which prescribes

$$D_k = \begin{cases} \theta, & \lambda_k^{(n)} > \theta \\ \lambda_k^{(n)}, & \lambda_k^{(n)} \le \theta \end{cases} . \tag{14}$$

Thus, it is possible to achieve

$$R(D_\theta) = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \frac{1}{2}\max\left\{0, \log\frac{\lambda_k^{(n)}}{\theta}\right\} , \tag{15}$$

$$D_\theta = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \min\{\lambda_k^{(n)}, \theta\} . \tag{16}$$

Since the covariance matrix $K_{XX}^{(n)}$ of a stationary Gaussian random process is Toeplitz, by the Toeplitz distribution theorem (see e.g. Berger p. 112), we can evaluate the limits:

$$R(D_\theta) = \int_{-1/2}^{1/2} \frac{1}{2}\max\left\{0, \log\frac{S_{XX}(e^{j2\pi f})}{\theta}\right\} df , \tag{17}$$

$$D_\theta = \int_{-1/2}^{1/2} \min\{S_{XX}(e^{j2\pi f}), \theta\} df , \tag{18}$$

It turns out this is optimal (see e.g. Berger). It is convenient to define

$$D(e^{j2\pi f}) = \begin{cases} \theta, & S_{XX}(e^{j2\pi f}) > \theta \\ S_{XX}(e^{j2\pi f}), & S_{XX}(e^{j2\pi f}) \le \theta \end{cases} , \tag{19}$$

after which the rate-distortion function can be written as

$$R(D_\theta) = \int_{-1/2}^{1/2} \frac{1}{2}\log\frac{S_{XX}(e^{j2\pi f})}{D(e^{j2\pi f})} df , \tag{20}$$

$$D_\theta = \int_{-1/2}^{1/2} D(e^{j2\pi f}) df . \tag{21}$$

V. Inverse Water Filling and Sampling

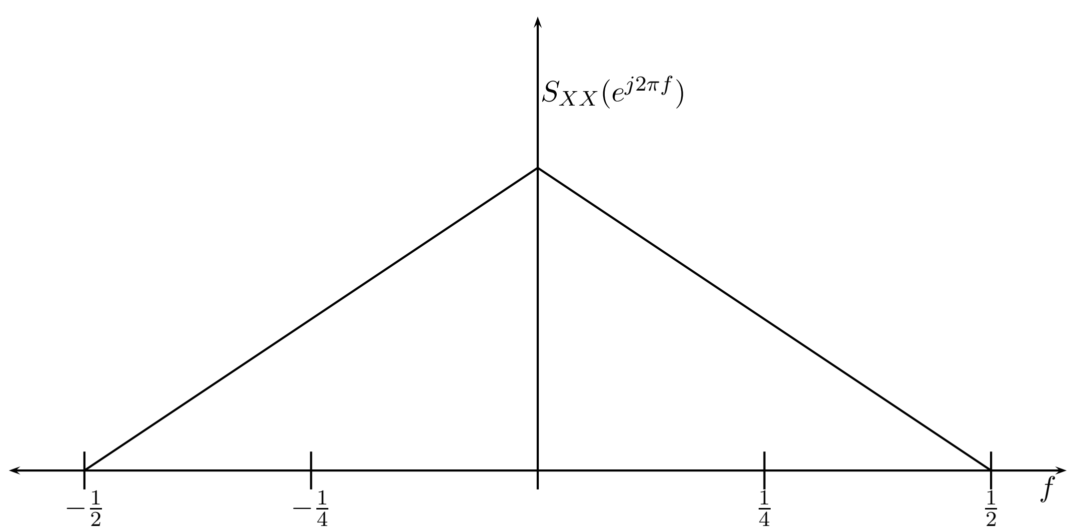

Fig. 3. Power-spectral density of discrete-time Gaussian random process $\ldots, X_{-1}, X_0, X_1, \ldots$.

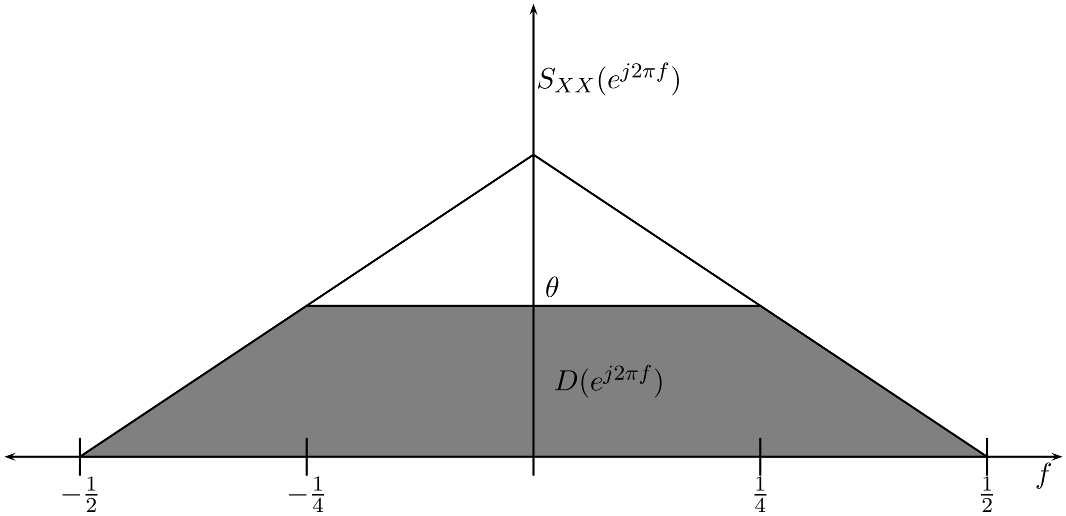

Fig. 4. Power-spectral density of discrete-time Gaussian random process $\ldots, X_{-1}, X_0, X_1, \ldots$.

The above result shows us something interesting. Even if the power spectrum is non-zero everywhere, it may still be possible to ignore frequencies if they fall below a certain threshold. Consider a power spectrum as in Figure 3. Figure 4 overlays the distortion spectrum $D(e^{j2\pi f})$ over this power spectral density. Let us construct a filter

$$H(e^{j2\pi f}) = \begin{cases} 1, & D(e^{j2\pi f}) = \theta \\ 0, & \text{otherwise} \end{cases} \quad . \tag{22}$$

By passing the original process through this filter, the power spectrum of the output process $\{\tilde{X}_n\}_{n=-\infty}^{\infty}$ would look as it does in Figure 5.

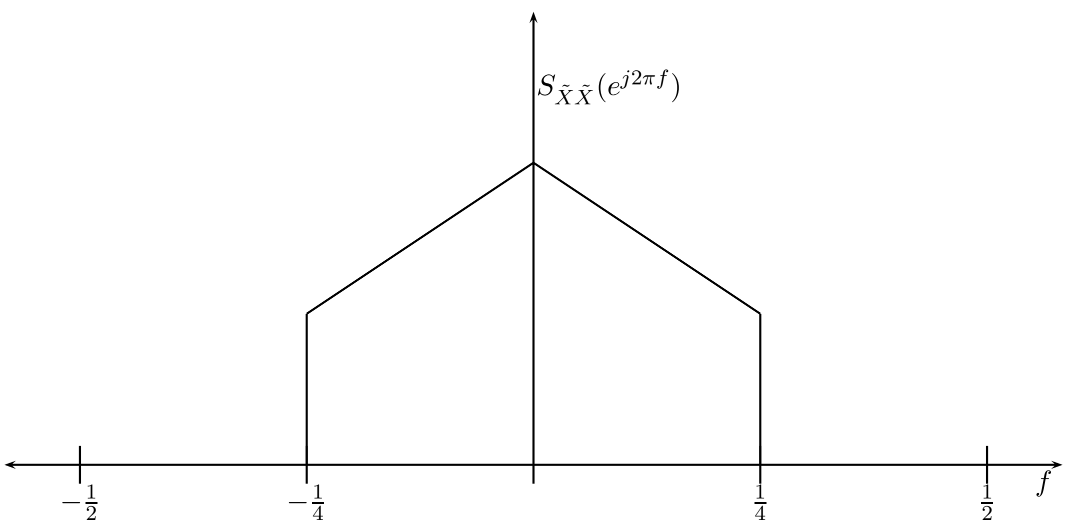

Fig. 5. Power-spectral density Gaussian random process after filtering by a distortion-equalizing filter.

Note that this output already has a distortion against the original process of

$$d_\theta = \int_{f:D(e^{j2\pi f})<\theta} D(e^{j2\pi f}) df \ . \tag{23}$$

However, by targeting distortion $D_\theta - d_\theta$ on the output process, the same rate allocations for θ distortion on the non-zero parts of the power spectrum, will result in the same rate-distortion tradeoff.

The connection to the sampling theorem can be seen by noting if the output process is bandlimited, one may be able to downsample. Specifically, if $S_{\tilde{X}\tilde{X}}(e^{j2\pi f}) = 0$ for $|f| > B$, it could be possible to downsample by

$M = \lfloor \frac{B}{2} \rfloor$, i.e. $Y_n = \tilde{X}_{Mn}$, which would then result in a power spectrum as in Figure 6. Note that from Section III, it is possible to reconstruct an $\tilde{X}_n$ from Y_n with 0 mean-squared error.

Now, by targeting the distortion $\frac{D_\theta - d_\theta}{M}$, which can be realized by allocation rates based on distortion $\frac{\theta}{M}$ for the non-zero parts of the spectrum, and following this up with an appropriate Wiener filter to upsample and lower pass filter the reconstruction, one can also achieve the optimal rate-distortion tradeoff. The key difference is that such an approach can reduce the number of samples to process by a factor of M. Indeed, to see any computational benefit in practice, $H(e^{j2\pi f})$ would have to be approximated (or realized) as an FIR filter, which would enable an efficient polyphase implementation.

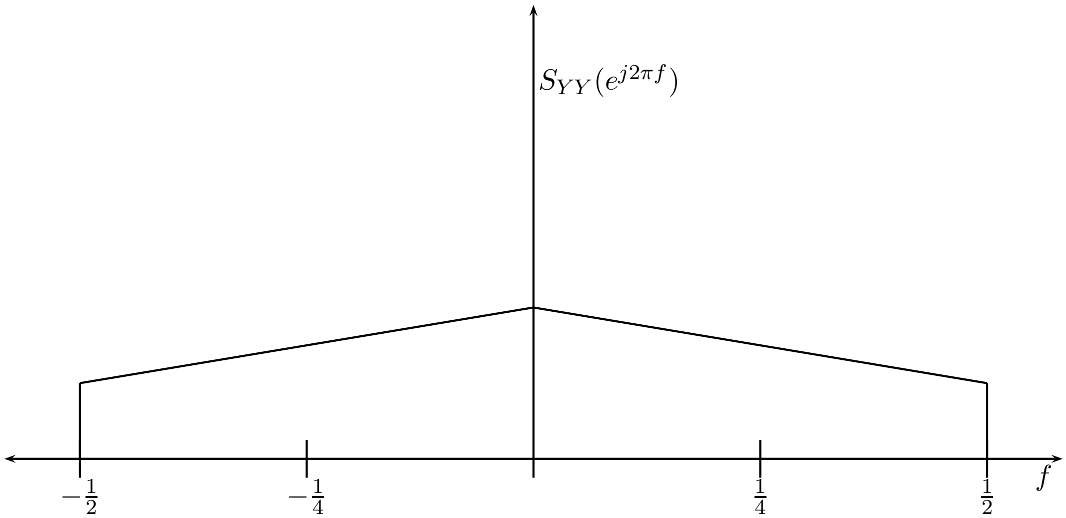

Fig. 6. Power-spectral density Gaussian random process after filtering by a distortion-equalizing filter.

Indeed, one can extend this analysis to consider scenarios in which certain bands have a power spectrum of 0, even if they might not be at the tails. Then, by appropriately constructing band-pass filters and modulating the outputs, the downsampling can be extended.

Appendix A
Relationship between CTFT and DTFT of a sampled signal

The following shows the derivation of (3). The relationship can be derived as follows:

$$
\begin{aligned}
X(e^{j2\pi f}) &= \sum_{n=-\infty}^{\infty} x[n] e^{-j2\pi f n} && (24)\\
&= \sum_{n=-\infty}^{\infty} x(nT) e^{-j2\pi \frac{f}{T} nT}\\
&= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(nT-t) x(t) e^{-j2\pi \frac{f}{T} t} dt\\
&= \int_{-\infty}^{\infty} \left(\sum_{n=-\infty}^{\infty} \delta(nT-t) \right) x(t) e^{-j2\pi \frac{f}{T} t} dt\\
&= \int_{-\infty}^{\infty} \left(\sum_{n=-\infty}^{\infty} \frac{1}{T} e^{-j2\pi n \frac{t}{T}} \right) x(t) e^{-j2\pi \frac{f}{T} t} dt && (25)\\
&= \frac{1}{T} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-j2\pi \frac{n}{T} t} x(t) e^{-j2\pi \frac{f}{T} t} dt\\
&= \frac{1}{T} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-j2\pi \frac{n}{T} t} x(t) e^{-j2\pi \frac{f}{T} t} dt\\
&= \frac{1}{T} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-j2\pi \frac{n}{T} t} x(t) e^{-j2\pi \frac{f}{T} t} dt\\
&= \frac{1}{T} \sum_{n=-\infty}^{\infty} \mathbf{X}\left(\frac{f-n}{T} \right) && (26)
\end{aligned}
$$

where (25) follows from the Fourier series of an impulse train.